

A CLASSIFICATION OF IRREDUCIBLE WAKIMOTO MODULES FOR THE AFFINE LIE ALGEBRA $A_1^{(1)}$

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ABSTRACT. By using methods developed in Adamović (Comm. Math. Phys. 270 (2007) 141-161) we study the irreducibility of certain Wakimoto modules for \widehat{sl}_2 at the critical level. We classify all $\chi \in \mathbb{C}((z))$ such that the corresponding Wakimoto module W_χ is irreducible. It turns out that zeros of Schur polynomials play important role in the classification result.

1. INTRODUCTION

In the representation theory of affine Kac-Moody Lie algebras, representations at the critical level belong to one of the most important cases. The Kac-Kazhdan conjecture for characters motivates explicit realizations of irreducible highest weight modules at the critical level. These representations can be realized by using Wakimoto modules (cf. [F], [FF1], [FF2], [FP], [S], [W]). In [A2] we introduced an infinite-dimensional Lie superalgebra \mathcal{A} which is a certain limit of $N=2$ superconformal algebras obtained by using Kazama-Suzuki mappings (cf. [A1], [FST], [KS]). We also constructed a family of functors which send irreducible \mathcal{A} -modules to irreducible modules for the affine Lie algebra $A_1^{(1)}$ at the critical level. By using this construction we proved irreducibility of a large family of Wakimoto modules W_χ parameterized by $\chi \in \mathbb{C}((z))$. In this paper we shall completely solve the irreducibility problem for Wakimoto modules W_χ . We shall describe all $\chi \in \mathbb{C}((z))$ such that W_χ is irreducible.

We first consider \mathcal{A} -modules \tilde{F}_χ constructed by using representations of the infinite-dimensional Clifford algebra and also parameterized by $\chi \in \mathbb{C}((z))$. The functor \mathcal{L}_0 sends \tilde{F}_χ to the Wakimoto module $W_{-\chi}$ (cf. [A2]). Then $W_{-\chi}$ is irreducible $A_1^{(1)}$ -module if and only if \tilde{F}_χ is irreducible \mathcal{A} -module (cf. Theorems 5.2 and 5.3). So we only need to classify $\chi \in \mathbb{C}((z))$ such that \tilde{F}_χ is irreducible. By combining results from [A2] and results from the present paper, we obtain the following classification result.

Theorem 1.1. *Assume that $\chi \in \mathbb{C}((z))$. Then the Wakimoto module $W_{-\chi}$ is an irreducible $A_1^{(1)}$ -module (resp. \tilde{F}_χ is irreducible \mathcal{A} -module) if and only if χ satisfies one of the following conditions:*

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(i) *There is $p \in \mathbb{Z}_{\geq 0}$, $p \geq 1$ such that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_p \neq 0.$$

(ii)

$$\chi(z) = \sum_{n=0}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z}).$$

(iii) *There is $\ell \in \mathbb{Z}_{>0}$ such that*

$$\chi(z) = \frac{\ell+1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

and $S_{\ell}(-\chi) \neq 0$, where $S_{\ell}(-\chi) = S_{\ell}(-\chi_{-1}, -\chi_{-2}, \dots)$ is a Schur polynomial.

We also prove that when the Wakimoto module $W_{-\chi}$ is reducible, then it contains an irreducible submodule.

Although the methods used in this paper can be mainly applied for the affine Lie algebra $A_1^{(1)}$, we believe that the main classification result can be extended for higher rank case. We hope to study this problem in our future publications.

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2. CLIFFORD VERTEX SUPERALGEBRAS

The Clifford algebra CL is a complex associative algebra generated by

$$\Psi^{\pm}(r), \quad r \in \frac{1}{2} + \mathbb{Z},$$

and relations

$$\{\Psi^{\pm}(r), \Psi^{\mp}(s)\} = \delta_{r+s,0}; \quad \{\Psi^{\pm}(r), \Psi^{\pm}(s)\} = 0$$

where $r, s \in \frac{1}{2} + \mathbb{Z}$.

Let F be the irreducible CL -module generated by the cyclic vector $\mathbf{1}$ such that

$$\Psi^{\pm}(r)\mathbf{1} = 0 \quad \text{for } r > 0.$$

As a vector space,

$$F = \bigwedge (\Psi^{-}(-n - \frac{1}{2}), n \in \mathbb{Z}_{\geq 0}) \otimes \bigwedge (\Psi^{+}(-n - \frac{1}{2}), n \in \mathbb{Z}_{\geq 0})$$

where $\bigwedge(x_i, i \in I)$ denotes the exterior algebra with generators $x_i, i \in I$.

Define the following fields on F

$$\Psi^{+}(z) = \sum_{n \in \mathbb{Z}} \Psi^{+}(n + \frac{1}{2}) z^{-n-1}, \quad \Psi^{-}(z) = \sum_{n \in \mathbb{Z}} \Psi^{-}(n + \frac{1}{2}) z^{-n-1}.$$

The fields $\Psi^+(z)$ and $\Psi^-(z)$ generate on F the unique structure of a simple vertex superalgebra (cf. [K2], [FB]).

Define the following Virasoro vector in F :

$$\omega^{(f)} = \frac{1}{2}(\Psi^+(-\frac{3}{2})\Psi^-(-\frac{1}{2}) + \Psi^-(-\frac{3}{2})\Psi^+(-\frac{1}{2}))\mathbf{1}.$$

Then the components of the field $L^{(f)}(z) = Y(\omega^{(f)}, z) = \sum_{n \in \mathbb{Z}} L^{(f)}(n)z^{-n-2}$ defines on F a representation of the Virasoro algebra with central charge 1.

Set

$$J^{(f)}(z) = Y(\Psi^+(-\frac{1}{2})\Psi^-(-\frac{1}{2})\mathbf{1}, z) = \sum_{n \in \mathbb{Z}} J^{(f)}(n)z^{-n-1}.$$

Then we have

$$[J^{(f)}(n), \Psi^\pm(m + \frac{1}{2})] = \pm \Psi^\pm(m + n + \frac{1}{2}).$$

Let $\tilde{F} = \text{Ker}_F \Psi^-(\frac{1}{2})$ be the subalgebra of the vertex superalgebra F generated by the fields

$$\partial \Psi^+(z) = \sum_{n \in \mathbb{Z}} -n \Psi^+(n - \frac{1}{2})z^{-n-1} \quad \text{and} \quad \Psi^-(z) = \sum_{n \in \mathbb{Z}} \Psi^-(n + \frac{1}{2})z^{-n-1}.$$

Then \tilde{F} is a simple vertex superalgebra and it is $\frac{1}{2}\mathbb{Z}_{\geq 0}$ -graded with respect to the operator $L^f(0)$. Let us describe the basis of \tilde{F} . A superpartition is a sequence $\lambda = (\lambda_n)_{n \in \mathbb{Z}_{>0}}$ in $S \cup \{0\}$, $S \subset \mathbb{Q}_+$, such that

$$\lambda_1 > \lambda_2 > \dots \quad \text{and} \quad \lambda_n = 0 \quad \text{for } n \text{ sufficiently large.}$$

Define the length of partition by $\ell(\lambda) = \max\{n \mid \lambda_n \neq 0\}$. If $\ell(\lambda) = \ell$ we write $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Let ϕ denotes the superpartition with all the entries being zero. Then we define $\ell(\phi) = 0$.

Let \mathcal{P} be the set of all superpartitions in $(\frac{1}{2} + \mathbb{Z}_{\geq 0}) \cup \{0\}$ and $\overline{\mathcal{P}}$ be the set of all superpartitions in $(\frac{3}{2} + \mathbb{Z}_{\geq 0}) \cup \{0\}$. Then we have

$$\mathcal{P} = \cup_{r=0}^{\infty} \mathcal{P}_r, \quad \overline{\mathcal{P}} = \cup_{r=0}^{\infty} \overline{\mathcal{P}}_r$$

where $\mathcal{P}_0 = \overline{\mathcal{P}}_0 = \{\phi\}$, and

$$\begin{aligned} \mathcal{P}_r &= \{\lambda = (\lambda_1, \dots, \lambda_r) \in (\frac{1}{2} + \mathbb{Z})^r \mid \lambda_1 > \lambda_2 > \dots > \lambda_r \geq 1/2\} \\ \overline{\mathcal{P}}_r &= \{\lambda = (\lambda_1, \dots, \lambda_r) \in (\frac{1}{2} + \mathbb{Z})^r \mid \lambda_1 > \lambda_2 > \dots > \lambda_r \geq 3/2\}. \end{aligned}$$

For $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathcal{P}_r$, $\mu = (\mu_1, \dots, \mu_s) \in \overline{\mathcal{P}}_s$ we set

$$\begin{aligned} v_{\lambda, \mu} &:= \Psi^-(-\lambda_1) \cdots \Psi^-(-\lambda_r) \Psi^+(-\mu_1) \cdots \Psi^+(-\mu_s) \mathbf{1} \\ v_{\lambda, \phi} &:= \Psi^-(-\lambda_1) \cdots \Psi^-(-\lambda_r) \mathbf{1}, \quad v_{\phi, \mu} := \Psi^+(-\mu_1) \cdots \Psi^+(-\mu_s) \mathbf{1}, \\ v_{\phi, \phi} &= \mathbf{1}. \end{aligned}$$

Then the set

$$(2.1) \quad \{v_{\lambda, \mu} \mid (\lambda, \mu) \in \mathcal{P} \times \overline{\mathcal{P}}\}$$

is a basis of \widetilde{F} .

3. THE VERTEX SUPERALGEBRA \mathcal{V} AND ITS MODULES

In this section we shall recall definition of the vertex superalgebra \mathcal{V} and certain results from [A2]. Let $M(0) = \mathbb{C}[\gamma^+(n), \gamma^-(n) \mid n < 0]$ be the commutative vertex algebra generated by the fields

$$\gamma^\pm(z) = \sum_{n < 0} \gamma^\pm(n) z^{-n-1}.$$

(cf. [F]). Let $\chi^\pm(z) = \sum_{n \in \mathbb{Z}} \chi_n^\pm z^{-n-1} \in \mathbb{C}((z))$. Let $M(0, \chi^+, \chi^-)$ denotes the 1-dimensional irreducible $M(0)$ -module with the property that every element $\gamma^\pm(n)$ acts on $M(0, \chi^+, \chi^-)$ as multiplication by $\chi_n^\pm \in \mathbb{C}$.

Let now \mathcal{F} be the vertex superalgebra generated by the fields $\Psi^\pm(z)$ and $\gamma^\pm(z)$. Therefore $\mathcal{F} = F \otimes M(0)$. As in [A2], denote by \mathcal{V} the vertex subalgebra of the vertex superalgebra \mathcal{F} generated by the following vectors

$$(3.2) \quad \tau^\pm = (\Psi^\pm(-\tfrac{3}{2}) + \gamma^\pm(-1)\Psi^\pm(-\tfrac{1}{2}))\mathbf{1},$$

$$(3.3) \quad j = \frac{\gamma^+(-1) - \gamma^-(-1)}{2}\mathbf{1},$$

$$(3.4) \quad \nu = \frac{2\gamma^+(-1)\gamma^-(-1) + \gamma^+(-2) + \gamma^-(-2)}{4}\mathbf{1}.$$

The vertex superalgebra structure on \mathcal{V} is generated by the following fields

$$(3.5) \quad G^\pm(z) = Y(\tau^\pm, z) = \sum_{n \in \mathbb{Z}} G^\pm(n + \tfrac{1}{2}) z^{-n-2},$$

$$(3.6) \quad S(z) = Y(\nu, z) = \sum_{n \in \mathbb{Z}} S(n) z^{-n-2},$$

$$(3.7) \quad T(z) = Y(j, z) = \sum_{n \in \mathbb{Z}} T(n) z^{-n-1}.$$

Denote by \mathcal{A} the Lie superalgebra with basis $S(n), T(n), G^\pm(r), C, n \in \mathbb{Z}, r \in \tfrac{1}{2} + \mathbb{Z}$ and (anti)commutation relations given by

$$\begin{aligned} [S(m), S(n)] &= [S(m), T(n)] = [S(m), G^\pm(r)] = 0, \\ [T(m), T(n)] &= [T(m), G^\pm(r)] = 0, \\ [C, S(m)] &= [C, T(n)] = [C, G^\pm(r)] = 0, \\ \{G^+(r), G^-(s)\} &= 2S(r+s) + (r-s)T(r+s) + \frac{C}{3}(r^2 - \tfrac{1}{4})\delta_{r+s,0}, \\ \{G^+(r), G^+(s)\} &= \{G^-(r), G^-(s)\} = 0 \end{aligned}$$

for all $n \in \mathbb{Z}, r, s \in \tfrac{1}{2} + \mathbb{Z}$.

By using the commutator formulae for vertex superalgebras, we have that the components of fields (3.5)-(3.7) satisfy the (anti)commutation relation for the Lie superalgebra \mathcal{A} such that the central element C acts as multiplication by $c = -3$. Let \mathcal{V}^{com} be the commutative vertex subalgebra of \mathcal{V} generated by the fields $T(z)$ and $S(z)$. Clearly, $\mathcal{V}^{com} \cong M_T(0) \otimes M_S(0)$, where $M_T(0)$ (resp. $M_S(0)$) is the subalgebra of \mathcal{V}^{com} generated by the field $T(z)$ (resp. $S(z)$).

Recall that \mathcal{V} admits the following \mathbb{Z} -graduation:

$$\begin{aligned} \mathcal{V} &= \bigoplus_{m \in \mathbb{Z}} \mathcal{V}^m \\ \mathcal{V}^m &= \text{span}_{\mathbb{C}} \{ G^+(-n_1 - \frac{3}{2}) \cdots G^+(-n_r - \frac{3}{2}) G^-(-k_1 - \frac{3}{2}) \cdots G^-(-k_s - \frac{3}{2}) w \mid \\ &\quad w \in \mathcal{V}^{com}, n_i, k_j \in \mathbb{Z}_{\geq 0}, r - s = m \}. \end{aligned}$$

Now we shall consider a family of irreducible \mathcal{V} -modules.

For $\chi^+, \chi^- \in \mathbb{C}((z))$ we set $F(\chi^+, \chi^-) := F \otimes M(0, \chi^+, \chi^-)$.

Then $F(\chi^+, \chi^-)$ is a module for the vertex superalgebra \mathcal{V} , and therefore for the Lie superalgebra \mathcal{A} .

Since $M(0, \chi^+, \chi^-)$ is one-dimensional, we have that as a vector space

$$(3.8) \quad F(\chi^+, \chi^-) \cong F \cong \bigwedge (\Psi^\pm(-i - \frac{1}{2}) \mid i \geq 0).$$

Now let $\chi(z) \in \mathbb{C}((z))$. Define:

$$\tilde{F}_\chi := \tilde{F} \otimes M(0, 0, \chi) \subset F(0, \chi).$$

The operator $J^f(0)$ acts semisimply on \tilde{F}_χ and it defines the following \mathbb{Z} -graduation

$$\tilde{F}_\chi = \bigoplus_{j \in \mathbb{Z}} \tilde{F}_\chi^j, \quad \tilde{F}_\chi^j = \{v \in \tilde{F}_\chi \mid J^f(0)v = jv\}.$$

The \mathcal{A} -module structure on \tilde{F}_χ is uniquely determined by the following action of the Lie superalgebra \mathcal{A} on \tilde{F} :

$$(3.9) \quad G^+(i - \frac{1}{2}) = -i\Psi^+(i - \frac{1}{2}),$$

$$(3.10) \quad G^-(i - \frac{1}{2}) = -i\Psi^-(i - \frac{1}{2}) + \sum_{k=-p}^{\infty} \chi_{-k} \Psi^-(k + i - \frac{1}{2}).$$

Now we shall first recall the following irreducibility result:

Proposition 3.1 ([A2], Proposition 5.2). *Assume that $p \in \mathbb{Z}_{\geq 0}$ and that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

satisfies the following conditions

$$(3.11) \quad \chi_p \neq 0,$$

$$(3.12) \quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z}) \text{ if } p = 0.$$

Then \tilde{F}_χ is an irreducible \mathcal{V} -module.

4. SCHUR POLYNOMIALS AND IRREDUCIBILITY OF \tilde{F}_χ

In this section we shall extend the irreducibility result from Proposition 3.1. We shall always assume that $\chi(z)$ has the form

$$(4.13) \quad \chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z)),$$

where $\ell \in \mathbb{Z}$. Then the \mathcal{A} -module structure on \tilde{F}_χ is uniquely determined by the following action of the Lie superalgebra \mathcal{A} on \tilde{F} :

$$(4.14) \quad G^+(i - \tfrac{1}{2}) = -i\Psi^+(i - \tfrac{1}{2}),$$

$$(4.15) \quad G^-(i - \tfrac{1}{2}) = (\ell + 1 - i)\Psi^-(i - \tfrac{1}{2}) + \sum_{n=1}^{\infty} \chi_{-n}\Psi^-(n + i - \tfrac{1}{2}).$$

By Proposition 3.1 we know that if ℓ is generic or $\ell = 0$, then \tilde{F}_χ is an irreducible module. We shall consider the case when $\ell \in \mathbb{Z}_{>0}$, and find a sufficient condition on $\chi(z)$ so that \tilde{F}_χ is irreducible.

For every $s \in \mathbb{Z}_{>0}$, we define

$$\Omega_s = \Psi^+(-s - \tfrac{1}{2})\Psi^+(-s + \tfrac{1}{2}) \cdots \Psi^+(-\tfrac{3}{2})\mathbf{1} \in \tilde{F}_\chi.$$

We shall need the following lemma. The proof will use only the action of the operators $G^+(i - \tfrac{1}{2})$, $i \in \mathbb{Z}$.

Lemma 4.1. *Assume that $U \subset \tilde{F}_\chi$ is any submodule, $U \neq \{0\}$. Then there is $s \in \mathbb{Z}_{>0}$ such that*

$$\Omega_s \in U.$$

Proof. For $\lambda \in \mathcal{P}$ and $t \in \overline{\mathcal{P}}$, we set

$$G_\lambda^+ = \begin{cases} 1 & \text{if } \lambda = \phi \\ G^+(\lambda_1) \cdots G^+(\lambda_r) & \text{if } \lambda = (\lambda_1, \dots, \lambda_r) \end{cases}$$

$$G_{-t}^+ = \begin{cases} 1 & \text{if } t = \phi \\ G^+(-t_1) \cdots G^+(-t_r) & \text{if } t = (t_1, \dots, t_r) \end{cases}$$

Let $v \in U$, $v \neq 0$. Then v has unique decomposition

$$v = \sum_{(\lambda, \mu) \in \mathcal{P} \times \overline{\mathcal{P}}} C_{\lambda, \mu} v_{\lambda, \mu} \quad (C_{\lambda, \mu} \in \mathbb{C})$$

in the basis (2.1). Let $\ell = \max\{\ell(\lambda) \mid C_{\lambda,\mu} \neq 0\}$. We can choose $(\bar{\lambda}, \bar{\mu}) \in \mathcal{P} \times \bar{\mathcal{P}}$ such that

- (1) $C_{\bar{\lambda}, \bar{\mu}} \neq 0, \ell(\bar{\lambda}) = \ell,$
- (2) $\ell(\bar{\mu}) = \ell_1 = \min\{\ell(\mu) \mid \mu \in T_1\}$ where $T_1 = \{\mu \in \bar{\mathcal{P}} \mid C_{\bar{\lambda}, \mu} \neq 0\}.$

If $T_1 = \{\phi\}$, we set $f = G_{\bar{\lambda}}^+$. Otherwise, let $s \in \mathbb{Z}_{>0}$ be such that

$$\max\{\mu_1 \mid \mu = (\mu_1, \dots, \mu_l) \in T_1\} = s + \frac{1}{2}.$$

If $\bar{\mu} = (\bar{\mu}_1, \dots, \bar{\mu}_{\ell_1})$ and $0 < \ell_1 = \ell(\bar{\mu}) < s$, there are unique $t_1 > \dots > t_p, p = s - \ell_1$, such that

$$\{t_1, \dots, t_p\} = \{\frac{3}{2}, \dots, s + \frac{1}{2}\} \setminus \{\bar{\mu}_1, \dots, \bar{\mu}_{\ell_1}\}.$$

Define now $t \in \bar{\mathcal{P}}$ in the following way:

$$t = \begin{cases} \phi & \text{if } \ell_1 = s \\ (s + 1/2, \dots, 3/2) & \text{if } \ell_1 = 0 \\ (t_1, \dots, t_p) & \text{if } 0 < \ell_1 < s \end{cases}.$$

Then we set

$$f = G_{-t}^+ G_{\bar{\lambda}}^+.$$

By construction we have that $G_{\bar{\lambda}}^+$ annihilates basis vectors $v_{\lambda,\mu}$ such that $\ell(\lambda) \leq \ell, \lambda \neq \bar{\lambda}$, and G_{-t}^+ annihilates all $v_{\bar{\lambda},\mu}$, where $\mu \in T_1 \setminus \{\bar{\mu}\}$. Therefore,

$$\begin{aligned} f v_{\lambda,\mu} &= 0 \quad \text{if } C_{\lambda,\mu} \neq 0 \text{ and } (\lambda, \mu) \neq (\bar{\lambda}, \bar{\mu}), \\ f v_{\bar{\lambda}, \bar{\mu}} &= \nu \Omega_s \quad (\nu \neq 0) \quad \text{if } T_1 \neq \{\phi\}, \\ f v_{\bar{\lambda}, \bar{\mu}} &= \nu_1 \mathbf{1} \quad (\nu_1 \neq 0) \quad \text{if } T_1 = \{\phi\}. \end{aligned}$$

The proof follows. □

In order to present new irreducibility criterion, we shall first recall the definition of Schur polynomials.

Define the Schur polynomials $S_r(x_1, x_2, \dots)$ in variables x_1, x_2, \dots by the following equation:

$$(4.16) \quad \exp \left(\sum_{n=1}^{\infty} \frac{x_n}{n} y^n \right) = \sum_{r=0}^{\infty} S_r(x_1, x_2, \dots) y^r.$$

We shall also use the following formula for Schur polynomials:

$$(4.17) \quad S_r(x_1, x_2, \dots) = \frac{1}{r!} \det \begin{pmatrix} x_1 & x_2 & \cdots & x_r \\ -r+1 & x_1 & x_2 & \cdots & x_{r-1} \\ 0 & -r+2 & x_1 & \cdots & x_{r-2} \\ 0 & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & x_1 \end{pmatrix}$$

Lemma 4.2. *We have*

$$G^-(\frac{1}{2}) \cdots G^-(\ell - \frac{1}{2}) \Omega_\ell = (-1)^\ell \ell! S_\ell(-\chi) \mathbf{1}$$

where $S_\ell(-\chi) = S_\ell(-\chi_{-1}, \dots, -\chi_{-\ell}, \dots)$.

Proof. By using action (4.15) we get:

$$\begin{aligned} & G^-(\frac{1}{2}) \cdots G^-(\ell - \frac{1}{2}) \Omega_\ell \\ = & (\ell \Psi^-(\frac{1}{2}) + \chi_{-1} \Psi^-(\frac{3}{2}) + \cdots + \chi_{-\ell} \Psi^-(\ell + \frac{1}{2})) \cdots \\ & (2 \Psi^-(\ell - \frac{3}{2}) + \chi_{-1} \Psi^-(\ell - \frac{1}{2}) + \chi_{-2} \Psi^-(\ell + \frac{1}{2})) (\Psi^-(\ell - \frac{1}{2}) + \chi_{-1} \Psi^-(\ell + \frac{1}{2})) \Omega_\ell \\ = & \det \begin{pmatrix} \chi_{-1} & \chi_{-2} & \cdots & \chi_{-\ell} \\ \ell - 1 & \chi_{-1} & \chi_{-2} & \cdots & \chi_{-\ell+1} \\ 0 & \ell - 2 & \chi_{-1} & \cdots & \chi_{-\ell+2} \\ 0 & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1 & \chi_{-1} \end{pmatrix} \\ = & (-1)^\ell \ell! S_\ell(-\chi_{-1}, \dots, -\chi_{-\ell}, \dots) \mathbf{1}. \end{aligned}$$

(Here we use elementary properties of determinants and formula (4.17) for Schur polynomials). \square

Proposition 4.1. *Assume that $\ell \in \mathbb{Z}_{>0}$,*

$$\chi(z) = \frac{\ell + 1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

such that

$$S_\ell(-\chi) \neq 0.$$

Then \tilde{F}_χ is an irreducible \mathcal{V} -module.

Proof. First we shall prove that the vacuum vector is a cyclic vector for the $U(\mathcal{A})$ -action, i.e.,

$$(4.18) \quad U(\mathcal{A}) \cdot \mathbf{1} = \tilde{F}.$$

Take an arbitrary basis element

$$(4.19) \quad v = \Psi^+(-n_1 - \frac{1}{2}) \cdots \Psi^+(-n_r - \frac{1}{2}) \Psi^-(-k_1 - \frac{1}{2}) \cdots \Psi^-(-k_s - \frac{1}{2}) \mathbf{1} \in \tilde{F},$$

where $n_i, k_i \in \mathbb{Z}_{\geq 0}$, $n_1 > n_2 > \cdots > n_r > 0$, $k_1 > k_2 > \cdots > k_s \geq 0$.

Let $N \in \mathbb{Z}_{\geq 0}$ such that $N \geq k_1$. By using (4.15) we get that

$$G^-(-N - \frac{1}{2}) \cdots G^-(-\frac{3}{2}) G^-(-\frac{1}{2}) \mathbf{1} = C \Psi^-(-N - \frac{1}{2}) \cdots \Psi^-(-\frac{3}{2}) \Psi^-(-\frac{1}{2}) \mathbf{1},$$

where

$$C = (\ell + 1)(\ell + 2) \cdots (\ell + N + 1)$$

So $C \neq 0$, and we have that

$$\Psi^-(N - \frac{1}{2}) \cdots \Psi^-(\frac{3}{2}) \Psi^-(\frac{1}{2}) \mathbf{1} \in U(\mathcal{A}).\mathbf{1}.$$

By using this fact and the action of elements $G^+(i - \frac{1}{2})$, $i \in \mathbb{Z}$, we obtain that $v \in U(\mathcal{A}).\mathbf{1}$. In this way we proved (4.18).

It is enough to prove that every vector $u \in \tilde{F}_\chi$ is cyclic. So let $U = U(\mathcal{A}).u$. By using Lemma 4.1 we have that there is $s \in \mathbb{Z}_{>0}$ such that $\Omega_s \in U$. Assume that $s > \ell$. Then clearly

$$(4.20) \quad G^-(\ell + \frac{3}{2}) \cdots G^-(s + \frac{1}{2}) \Omega_s = C_1 \Omega_\ell$$

for certain non-zero constant C_1 . Similarly, if $s < \ell$ we see that

$$(4.21) \quad G^+(\ell + \frac{1}{2}) \cdots G^+(s + \frac{3}{2}) \Omega_s = C_2 \Omega_\ell, \quad (C_2 \neq 0).$$

Therefore we conclude that $\Omega_\ell \in U$.

Applying Lemma 4.2 we get

$$G^-(\frac{1}{2}) \cdots G^-(\ell - \frac{1}{2}) \Omega_\ell = \nu \mathbf{1}, \quad (\nu \neq 0).$$

Thus $\mathbf{1} \in U = U(\mathcal{A}).u$. Now relation (4.18) gives that u is a cyclic vector in \tilde{F}_χ . The proof follows. \square

Proposition 4.2. Assume that $\ell \in \mathbb{Z}_{>0}$ and $S_\ell(-\chi) = 0$.

- (i) Then $U_\chi = U(\mathcal{A}).\Omega_\ell$ is a proper submodule of \tilde{F}_χ . In particular, \tilde{F}_χ is reducible.
- (ii) U_χ is an irreducible \mathcal{V} -module.

Proof. Assume that $S_\ell(-\chi) = 0$. Define

$$\begin{aligned} w &= G^-(\frac{3}{2}) \cdots G^-(\ell - \frac{1}{2}) \Omega_\ell \\ &= ((-1)^{\ell-1} (\ell-1)! \Psi^+(-\ell - \frac{1}{2}) + a_1 \Psi^+(-\ell + \frac{1}{2}) + \cdots + a_{\ell-1} \Psi^+(-\frac{3}{2})) \mathbf{1} \end{aligned}$$

where $a_1, \dots, a_{\ell-1}$ are certain complex numbers.

Therefore $w \neq 0$. By using Lemma 4.2, the assumption $S_\ell(-\chi) = 0$ and the definition of w we get

$$G^\pm(n - \frac{1}{2})w = 0 \quad \text{for } n \in \mathbb{Z}_{>0}.$$

One can easily show that

$$G^+(-\ell + \frac{1}{2}) \cdots G^+(-\frac{3}{2})w = C \Omega_\ell \quad (C \neq 0),$$

which implies that $U_\chi = U(\mathcal{A}).w$. Every element of U_χ is a linear combination of vectors

$$(4.22) \quad G^-(n_1 - \frac{1}{2}) \cdots G^-(n_r - \frac{1}{2}) G^+(-m_1 - \frac{1}{2}) \cdots G^+(-m_s - \frac{1}{2})w,$$

for $n_i, m_i \in \mathbb{Z}_{\geq 0}$, $n_1 > n_2 > \cdots > n_r$, $m_1 > m_2 > \cdots > m_s$. But a vector (4.22) is either zero (if $G^+(-m_1 - \frac{1}{2}) \cdots G^+(-m_s - \frac{1}{2})w = 0$) or has the following non-trivial summand of lowest degree in \tilde{F} (with respect to $L^f(0)$)

$$C \Psi^-(n_1 - \frac{1}{2}) \cdots \Psi^-(n_r - \frac{1}{2}) \Psi^+(-m_1 - \frac{1}{2}) \cdots \Psi^+(-m_s - \frac{1}{2})w$$

where $C \neq 0$. From this one gets that $1 \notin U_\chi$. Therefore \tilde{F}_χ is a reducible module with the proper submodule U_χ . This proves assertion (i).

Assume now that $U \subset U_\chi$ is a non-zero submodule. Then Lemma 4.1 implies that there is $s \in \mathbb{Z}_{\geq 0}$ such that $\Omega_s \in U$. By using relations (4.20) and (4.21) from the proof of Proposition 4.1 we see that $\Omega_\ell \in U$. Therefore $U = U(\mathcal{A})\Omega_\ell = U_\chi$ and U_χ is an irreducible \mathcal{A} -module. This proves assertion (ii). \square

Proposition 4.3. *Assume that $\ell \in \mathbb{Z}$, $\ell < 0$.*

- (i) \tilde{F}_χ is reducible and $J_\chi = U(\mathcal{A}).1$ is its proper submodule.
- (ii) J_χ is an irreducible \mathcal{V} -module.

Proof. Let $q = -\ell - 1$. Then

$$G(n - \frac{1}{2}) = -(q + n)\Psi^-(n - \frac{1}{2}) + \sum_{n=1}^{\infty} \chi_{-n}\Psi^-(n + i - \frac{1}{2}).$$

By using similar arguments as in the proof of Proposition 4.2 one can see that $\Psi^-(-q - \frac{1}{2})1 \notin J_\chi$ which gives reducibility of \tilde{F}_χ . The proof that submodule J_χ is irreducible is completely analogous to that of Proposition 4.2 (ii). \square

Note that U_χ and J_χ are \mathbb{Z} -graded \mathcal{V} -modules with respect to $J^f(0)$:

$$(4.23) \quad U_\chi = \bigoplus_{i \in \mathbb{Z}} U_\chi^i, \quad U_\chi^i = \{v \in U_\chi \mid J^f(0)v = iv\},$$

$$(4.24) \quad J_\chi = \bigoplus_{i \in \mathbb{Z}} J_\chi^i, \quad J_\chi^i = \{v \in J_\chi \mid J^f(0)v = iv\}.$$

Now we are able to classify $\chi \in \mathbb{C}((z))$ such that \tilde{F}_χ is irreducible. We have proved the following classification result.

Theorem 4.1. *Assume that $\chi \in \mathbb{C}((z))$. Then the \mathcal{V} -module \tilde{F}_χ is irreducible if and only if χ satisfies one of the following conditions:*

- (i) *There is $p \in \mathbb{Z}_{\geq 0}$, $p \geq 1$ such that*

$$\chi(z) = \sum_{n=-p}^{\infty} \chi_{-n}z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_p \neq 0.$$

- (ii)

$$\chi(z) = \sum_{n=0}^{\infty} \chi_{-n}z^{n-1} \in \mathbb{C}((z)) \quad \text{and} \quad \chi_0 \in \{1\} \cup (\mathbb{C} \setminus \mathbb{Z}).$$

(iii) There is $\ell \in \mathbb{Z}_{\geq 0}$ such that

$$\chi(z) = \frac{\ell+1}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1} \in \mathbb{C}((z))$$

and $S_{\ell}(-\chi) \neq 0$.

5. WAKIMOTO MODULES

We shall first recall the definition of the Wakimoto modules at the critical level (cf. [F], [W]).

The Weyl vertex algebra W is generated by the fields

$$a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}, \quad a^*(z) = \sum_{n \in \mathbb{Z}} a^*(n) z^{-n},$$

whose components satisfy the commutation relations for the infinite-dimensional Weyl algebra

$$[a(n), a(m)] = [a^*(n), a^*(m)] = 0, \quad [a(n), a^*(m)] = \delta_{n+m, 0}.$$

Assume that $\chi(z) \in \mathbb{C}((z))$.

On the vertex algebra W exists the structure of the $A_1^{(1)}$ -module at the critical level defined by

$$\begin{aligned} e(z) &= a(z), \\ h(z) &= -2 : a^*(z) a(z) : - \chi(z) \\ f(z) &= - : a^*(z)^2 a(z) : - 2 \partial_z a^*(z) - a^*(z) \chi(z). \end{aligned}$$

This module is called the Wakimoto module and it is denoted by $W_{-\chi}$.

Let F_{-1} be the lattice vertex superalgebra V_L associated to the lattice $L = \mathbb{Z}\beta$, where $\langle \beta, \beta \rangle = -1$ (cf. [A1], [K2], [LL]). Then F_{-1} has the following \mathbb{Z} -gradation (cf. [A2]):

$$F_{-1} = \bigoplus_{j \in \mathbb{Z}} F_{-1}^j, \quad F_{-1}^j = \{v \in F_{-1} \mid \beta(0)v = -jv\}.$$

In [A2], we constructed mappings \mathcal{L}_s , $s \in \mathbb{Z}$, from the category of \mathcal{V} -modules to the category of $A_1^{(1)}$ -modules at the critical level. Let $V_{-2}(sl_2)$ denotes the universal affine vertex algebra for $A_1^{(1)}$ at the critical level, and $M_T(0)$ be the commutative subalgebra of \mathcal{V} generated by the field $T(z)$.

Theorem 5.1 ([A2], Theorem 6.2). *Assume that U is a \mathcal{V} -module which admits the following graduation:*

$$U = \bigoplus_{j \in \mathbb{Z}} U^j, \quad \mathcal{V}^i \cdot U^j \subset U^{i+j}.$$

Then

$$U \otimes F_{-1} = \bigoplus_{s \in \mathbb{Z}} \mathcal{L}_s(U) \quad \mathcal{L}_s(U) = \bigoplus_{i \in \mathbb{Z}} U^i \otimes F_{-1}^{-s+i}$$

and each $\mathcal{L}_s(U)$ is an $V_{-2}(sl_2) \otimes M_T(0)$ -module. If U is irreducible, then $\mathcal{L}_s(U)$ is an irreducible $A_1^{(1)}$ -module at the critical level.

In particular, the map \mathcal{L}_0 sends \mathcal{V} -module \tilde{F}_χ to the Wakimoto module $W_{-\chi}$ and

$$W_{-\chi} \cong \mathcal{L}_0(\tilde{F}_\chi) = \bigoplus_{j \in \mathbb{Z}} \tilde{F}_\chi^j \otimes F_{-1}^j.$$

Recall first:

Theorem 5.2. ([A2]) Assume that \tilde{F}_χ is an irreducible \mathcal{V} -module. Then $W_{-\chi}$ is irreducible $A_1^{(1)}$ -module at the critical level.

In the case of Wakimoto modules the converse is also true.

Theorem 5.3. Assume that \tilde{F}_χ is reducible. Then the Wakimoto module $W_{-\chi}$ is also reducible.

Proof. Assume that $N \subset \tilde{F}_\chi$ is any proper submodule. Take $s \in \mathbb{Z}_{>0}$ such that $\Omega_s \in N$ (cf. Lemma 4.1) and define $U = U(\mathcal{A})\Omega_s \subseteq N$. Then U admits the \mathbb{Z} -gradation

$$U = \bigoplus_{j \in \mathbb{Z}} U^j$$

where

$$U^j = \{v \in U \mid J^f(0)v = jv\} \subset \tilde{F}_\chi^j.$$

Then by using Theorem 5.1 we conclude that

$$\mathcal{L}_0(U) = \bigoplus_{j \in \mathbb{Z}} U^j \otimes F_{-1}^j$$

is an $A_1^{(1)}$ -module and it is a proper submodule of the Wakimoto module $W_{-\chi}$. The proof follows. \square

Corollary 5.1. The Wakimoto module $W_{-\chi}$ is irreducible if and only if $\chi(z) \in \mathbb{C}((z))$ satisfies one of the conditions (i)-(iii) of Theorem 4.1.

In the case when the module $W_{-\chi}$ is reducible, it contains irreducible submodule.

Corollary 5.2. Let $\chi(z) = \frac{\ell+1}{z} + \sum_{n=1}^{\infty} \chi_{-n}z^{n-1}$ and $\ell \in \mathbb{Z}$.

(i) Assume that $\ell \in \mathbb{Z}_{>0}$ and $S_\ell(-\chi) = 0$. Then

$$\mathcal{L}_0(U_\chi) = \bigoplus_{i \in \mathbb{Z}} U_\chi^i \otimes F_{-1}^i$$

is an irreducible submodule of $W_{-\chi}$.

(ii) Assume that $\ell < 0$. Then

$$\mathcal{L}_0(J_\chi) = \bigoplus_{i \in \mathbb{Z}} J_\chi^i \otimes F_{-1}^i$$

is an irreducible submodule of $W_{-\chi}$.

Proof. Propositions 4.2 and 4.3 imply that U_χ and J_χ are irreducible \mathcal{V} -modules which are \mathbb{Z} graded with graduations (4.23) and (4.24). Then Theorem 5.1 implies that $\mathcal{L}_0(U_\chi)$ and $\mathcal{L}_0(J_\chi)$ are irreducible $A_1^{(1)}$ -modules. The proof follows. \square

Remark 5.1. *In the case of reducible Wakimoto modules from Corollary 5.2 one can consider the action of sl_2 on $W_{-\chi}$ and the maximal sl_2 -integrable submodule $W_{-\chi}^{int}$ of $W_{-\chi}$. It is clear that $W_{-\chi}^{int}$ is a proper $A_1^{(1)}$ -submodule of $W_{-\chi}$. By combining our results and the results from [FG] and [ACM] one can easily show that $\mathcal{L}_0(U_\chi) = W_{-\chi}^{int}$ when $\ell > 0$ (resp. $\mathcal{L}_0(J_\chi) = W_{-\chi}^{int}$ when $\ell < 0$). So our method shows that the maximal integrable submodule of the Wakimoto module $W_{-\chi}$ is irreducible \widehat{sl}_2 -module at the critical level.*

Remark 5.2. *It is interesting to look at the case $\ell = 1$ and $\chi(z) = \frac{2}{z} + \sum_{n=1}^{\infty} \chi_{-n} z^{n-1}$. Then the Wakimoto module $W_{-\chi}$ is irreducible if $\chi_{-1} \neq 0$ and reducible if $\chi_{-1} = 0$. The reducible Wakimoto modules $W_{-\chi}$ such that*

$$\chi(z) = \frac{2}{z} + \sum_{n=2}^{\infty} \chi_{-n} z^{n-1}$$

(i.e., $\chi_{-1} = 0$) were studied in [FFR].

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